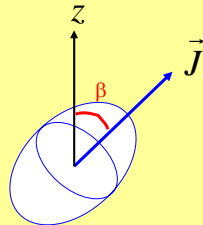Assume that the charge distribution is an ellipsoid of revolution with symmetry axis along the total angular momentum vector J:

As for the magnetic dipole moment, we specify the **intrinsic** electric quadrupole moment as the expectation value when J is maximally aligned with z:

$$Q_{\text{int}} = \left\langle \hat{E}_2 \right\rangle \Big|_{m_J = J}$$

"Quantum geometry": $\cos \beta = \dfrac{m_J}{|J|} = \dfrac{J}{\sqrt{J(J+1)}}$

If an electric field gradient is applied along the z-axis as shown, the observable energy will shift by an amount corresponding to the intrinsic quadrupole moment transformed to a coordinate system rotated through angle β to align with the z-axis

---

Result: 2

Let Q_{lab} be the electric quadrupole moment we **measure** for the ellipsoidal charge distribution with $m_J = J$:

$$Q_{\text{lab}} = \frac{1}{2}\left(3\cos^2 \beta - 1\right) Q_{\text{int}} = \left(\frac{J - 1/2}{J+1}\right) Q_{\text{int}}$$

standard, classical prescription for rotated coordinates

"Quantum geometry" for the rotation function

How do we apply this to anything?

1. A spherically symmetric state (L = 0) has Q_{int} = 0   (e.g. deuteron S-state)

2. Even a distorted state with J = ½ will not have an observable quadrupole moment

3. J = 1, L = 2 is the smallest value of total angular momentum for which we can observe a nonzero quadrupole moment

→ **these are the quantum numbers for the deuteron D-state!**

recall our model for the deuteron wave function:

$$\left| \psi_d \right\rangle = a \left| {}^3S_1 \right\rangle + b \left| {}^3D_1 \right\rangle$$

result for the quadrupole moment:

note cancellation here

$$Q_{\text{int}} = \frac{\sqrt{2}}{10} \left| a^*b \right| \left\langle r^2 \right\rangle_{SD} - \frac{1}{20} b^2 \left\langle r^2 \right\rangle_{DD}$$

$$= + 0.00286 \pm 0.00003 \text{ bn}$$

A good model of the N-N interaction can fit both the magnetic moment and the quadrupole moment of the deuteron with the same values of a and b! →

---

1. Independent of the value of L, the state with intrinsic spins coupled to S = 1 has lower energy

    → this implies a term proportional to:

    S=1 has lower energy

    $$- \left\langle \vec{S}_1 \bullet \vec{S}_2 \right\rangle = - \tfrac{1}{2} \left\langle S^2 - S_1^2 - S_2^2 \right\rangle = \begin{cases} -1/4, & S = 1 \\ +3/4, & S = 0 \end{cases}$$

2. The deuteron quadrupole moment implies a non-central component, i.e. the potential is not spherically symmetric. Since the symmetry axis for Q is along J, Q > 0 means that the matter distribution is stretched out along the J – axis:
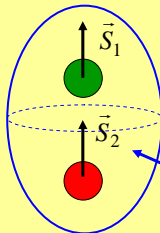
    → This implies a "tensor" force, proportional to:

    $\vec{S}_1$

    $\vec{S}_2$

    $$- \left\langle S_{12} \right\rangle = - \left\langle 3 \frac{(\vec{S}_1 \bullet \vec{r})(\vec{S}_2 \bullet \vec{r})}{r^2} - \vec{S}_1 \bullet \vec{S}_2 \right\rangle$$

    Q > 0, observed

    compare: magnetic dipole-dipole interaction, see Griffiths problem 6.20

3. There is also a spin-orbit term, as deduced from N-N scattering experiments with a polarized beam:

$$V_{S-O} \ \sim \ \left\langle \vec{L} \bullet \vec{S} \right\rangle$$

(This plays a very important role also in determining the correct order of energy levels in nuclear spectra – more later!)

4. Finally, all contributions to the N-N interaction are based on a microscopic meson exchange mechanism:
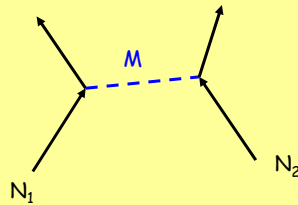
Where M is a π, ρ, ω ... meson, etc.

and each term has a spatial dependence of the form:

$$V(r) \ = \ g \ \frac{e^{-m\,r}}{r} \times (spin \ function)$$

PHYSICS REPORTS (Review Section of Physics Letters) 149, No. 1 (1987) 1–89. North-Holland, Amsterdam

(89 page exposition of one of only ~3 state-of-the-art models of the N-N interaction worldwide – constantly refined and updated since first release.)

THE BONN MESON-EXCHANGE MODEL FOR THE NUCLEON–NUCLEON INTERACTION*

R. MACHLEIDT[+]

Los Alamos National Laboratory, MS H850, Los Alamos, NM 87545, U.S.A.

and

Department of Physics, University of California, Los Angeles, CA 90024, U.S.A.

and

K. HOLINDE[++] and Ch. ELSTER

Institut für Theoretische Kernphysik der Universität Bonn, Nussallee 14–16, D-5300 Bonn, West Germany

pseudoscalar mesons:

$$V_{ps}(m_{ps}, r) = \frac{1}{12} \frac{g_{ps}^2}{4\pi} m_{ps}\left[\left(\frac{m_{ps}}{m}\right)^2 Y(m_{ps}r)\, \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + Z(m_{ps}r)\, S_{12}\right];$$ (F.6)

(tensor operator)

(they use "σ" for spin)

scalar mesons:

$$V_s(m_s, r) = -\frac{g_s^2}{4\pi} m_s\left\{\left[1 - \frac{1}{4}\left(\frac{m_s}{m}\right)^2\right]Y(m_s r) + \frac{1}{4m^2}\left[\nabla^2 Y(m_s r) + Y(m_s r)\nabla^2\right] + \frac{1}{2}Z_1(m_s r)\boldsymbol{L}\cdot\boldsymbol{S}\right\};$$ (F.7)

(spin-orbit interaction)

vector mesons:

$$V_v(m_v, r) = \frac{g_v^2}{4\pi} m_v\left\{\left[1 + \frac{1}{2}\left(\frac{m_v}{m}\right)^2\right]Y(m_v r) - \frac{3}{4m^2}\left[\nabla^2 Y(m_v r) + Y(m_v r)\nabla^2\right]\right.$$
$$+ \frac{1}{6}\left(\frac{m_v}{m}\right)^2 Y(m_v r)\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 - \frac{3}{2}Z_1(m_v r)\boldsymbol{L}\cdot\boldsymbol{S} - \frac{1}{12}Z(m_v r)S_{12}\right\}$$
$$+ \frac{1}{2}\frac{g_v f_v}{4\pi} m_v\{(m_v/m)^2\, Y(m_v r) + \frac{2}{3}(m_v/m)^2\, Y(m_v r)\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$$
$$- 4Z_1(m_v r)\boldsymbol{L}\cdot\boldsymbol{S} - \frac{1}{3}Z(m_v r)S_{12}\}$$
$$+ \frac{f_v^2}{4\pi} m_v\{\frac{1}{6}(m_v/m)^2\, Y(m_v r)\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 - \frac{1}{12}Z(m_v r)S_{12}\},$$

---

with

$$Y(x) = e^{-x}/x\,, \qquad Z(x) = (m_\alpha/m)^2(1 + 3/x + 3/x^2)Y(x)\,,$$ (F.8)
$$Z_1(x) = \left(\frac{m_\alpha}{m}\right)^2 (1/x + 1/x^2)Y(x)\,, \qquad S_{12} = 3\frac{(\boldsymbol{\sigma}_1 \cdot r)(\boldsymbol{\sigma}_2 \cdot r)}{r^2} - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2\,,$$

"Yukawa functions" and derivatives...

and

$$\nabla^2 = +\frac{1}{r}\frac{d^2}{dr^2}r - \frac{L^2}{r^2}\,.$$

We use units such that $\hbar = c = 1$ ($\hbar c = 197.3286$ MeV fm). The use of the form factor, eq. (3.3), at each vertex (with $n_\alpha = 1$) leads to the following extended expressions:

$$V_\alpha(r) = V_\alpha(m_\alpha, r) - \frac{\Lambda_{\alpha,2}^2 - m_\alpha^2}{\Lambda_{\alpha,2}^2 - \Lambda_{\alpha,1}^2} V_\alpha(\Lambda_{\alpha,1}, r) + \frac{\Lambda_{\alpha,1}^2 - m_\alpha^2}{\Lambda_{\alpha,2}^2 - \Lambda_{\alpha,1}^2} V_\alpha(\Lambda_{\alpha,2}, r)\,,$$ (F.9)

where $\Lambda_{\alpha,1} = \Lambda_\alpha + \varepsilon$, $\Lambda_{\alpha,2} = \Lambda_\alpha - \varepsilon$, $\varepsilon/\Lambda_\alpha \ll 1$. $\varepsilon = 10$ MeV is an appropriate choice. The full NN potential is the sum of the contributions from six mesons:

$$V(r) = \sum_{\alpha = \pi,\rho,\eta,\omega,\delta,\sigma} V_\alpha(r)$$

4

R. Machleidt et al., *The Bonn meson-exchange model for the nucleon–nucleon interaction*

Table 14
Meson and low-energy parameters (LEP) for the configuration space one-boson-exchange potential (OBEPR)

	$g_\alpha^2/4\pi; [f_\alpha/g_\alpha]$	m_α (MeV)	Λ_α (GeV)
π	14.9	138.03	1.3
ρ	0.95; [6.1]	769	1.3
η	3	548.8	1.5
ω	20; [0.0]	782.6	1.5
δ	2.6713	983	2.0
σ	7.7823 [a]	550 [a]	2.0

deuteron properties:

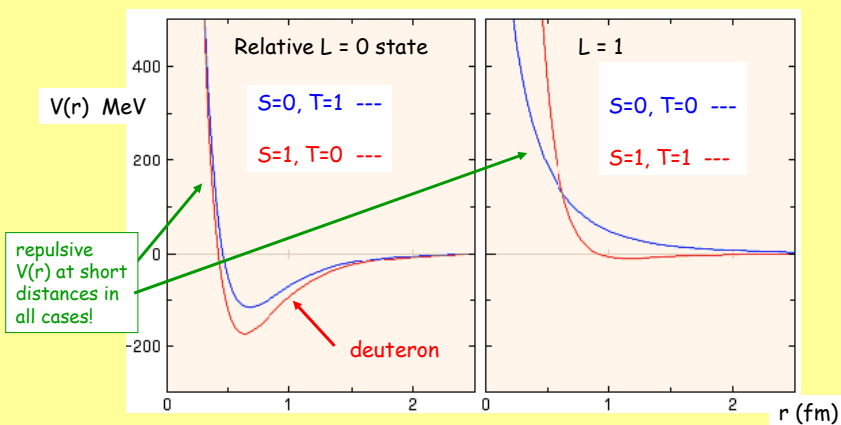| LEP	Theory
ε_d (MeV)	2.2246
P_D (%)	4.81
Q_d (fm^2)	0.274
μ_d (μ_N)	0.8524
A_S (fm$^{-1/2}$)	0.8860
D/S	0.0260
r_d (fm)	1.9691
a_s (fm)	−23.751
r_s (fm)	2.662
a_t (fm)	5.423
r_t (fm)	1.759

Impressively good agreement for ~ 10,000 experimental data points in assorted n-p and p-p scattering experiments plus deuteron observables: $\chi^2/\text{d.f} < \sim 1.07$!

low energy scattering parameters, etc:

It looks different in different spectroscopic states of the 2N system!

V(r) MeV

Relative L = 0 state

S=0, T=1 ---

S=1, T=0 ---

L = 1

S=0, T=0 ---

S=1, T=1 ---

repulsive V(r) at short distances in all cases!

deuteron

r (fm)

Only the deuteron is bound! Its quantum numbers have the deepest potential well.

5

On to Finite Nuclei!

Review:

1. Nuclear isotope chart: *(lecture 1)*

- 304 isotopes with $t_{\frac{1}{2}}$ > 10^9 yrs
               (age of the earth)

- 177 have even-Z, even-N and J^π = 0$^+$

- 121 are even-odd and **only 6 are odd-odd**

- N ≈ Z for light nuclei and N > Z for
       heavy nuclei
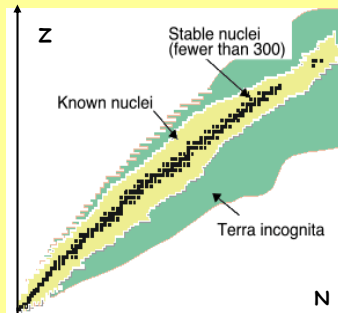
2. Elastic scattering of electrons: *(lecture 7)*

- nuclei are approximately spherical

- RMS charge radius $R = 1.2\ A^{1/3}\ \text{fm}$ fitted to electron scattering data

- mass ~ A, and radius ~ A$^{1/3}$ so the density M/V ≈ constant for nuclei
    (≈ 2 x 10^{17} kg/m^3), implying that nuclear matter is like an incompressible fluid

---

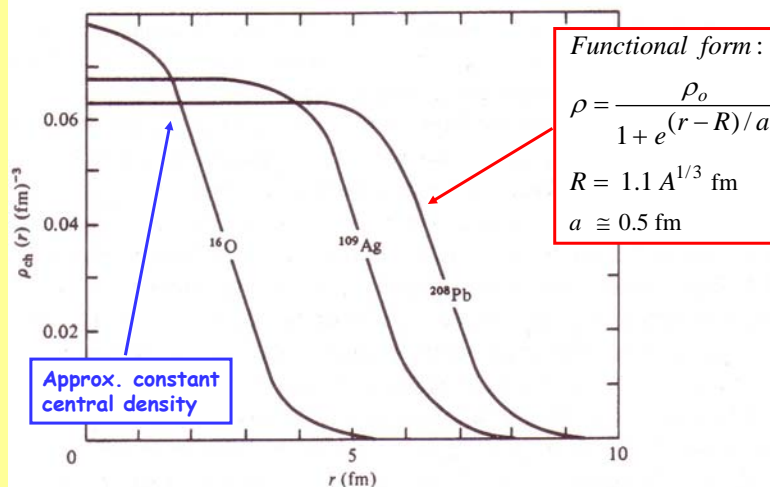Recall from lecture 7 -- Nuclear charge distributions from experiment: 13

$$Functional\ form:$$

$$\rho = \frac{\rho_o}{1 + e^{(r-R)/a}}$$

$$R = 1.1\ A^{1/3}\ \text{fm}$$

$$a \cong 0.5\ \text{fm}$$

Approx. constant central density

Fig. 4.3 The electric charge density of three nuclei as fitted by $\rho_{ch}(r) = \rho_{ch}^0/[1 + \exp((r - R)/a)]$. The parameters are taken from the compilation in Barrett, R. C. & Jackson, D. F. (1977), *Nuclear Sizes and Structure*, Oxford: Clarendon Press.

3. Inelastic electron scattering: *(lecture 9)*

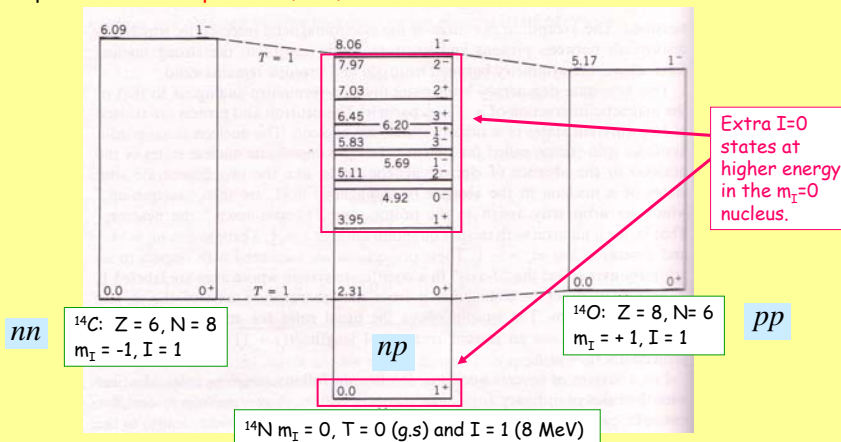• Excited states can be identified, on a scale of a few MeV above the ground state, e.g.
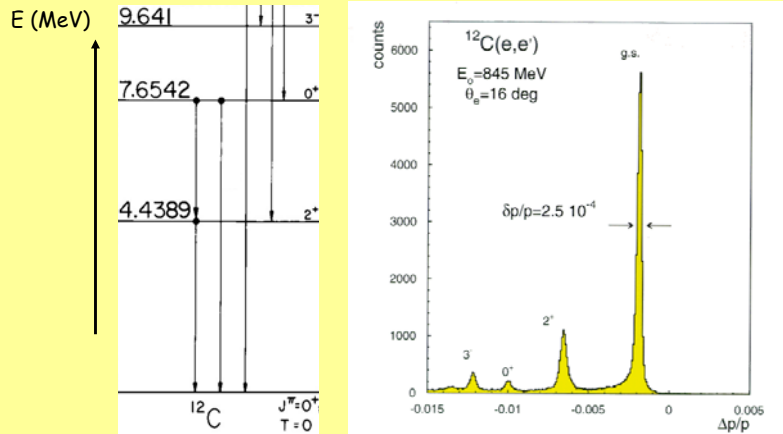
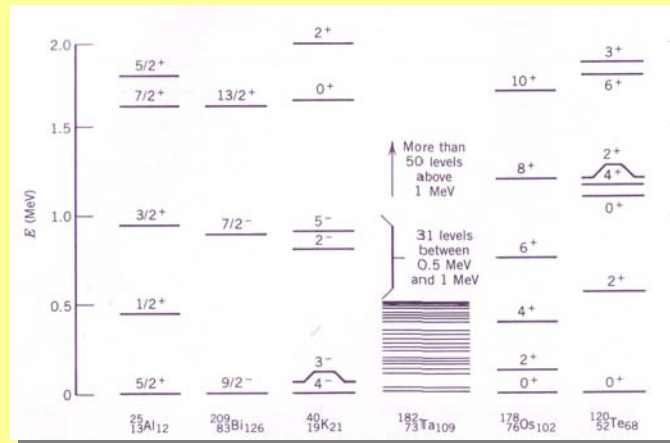
4. Quantum numbers for nuclear states: 14

• total angular momentum J, parity π

• isospin, I: *(lecture 13)*

     for a **nucleus**, $m_I = \frac{1}{2}$ (Z-N) and I = $|m_I|$, ie lowest energy has smallest I

• Example: "isobaric triplet" ^{14}C, ^{14}N, ^{14}O:

Extra I=0 states at higher energy in the m_I=0 nucleus.

nn ^{14}C: Z = 6, N = 8 m_I = -1, I = 1

np

^{14}O: Z = 8, N= 6 m_I = + 1, I = 1   *pp*

^{14}N m_I = 0, T = 0 (g.s) and I = 1 (8 MeV)

- states have integer or half-integer J depending on whether A is even or odd
- different systematics and energy level spacings for different nuclei
- some nuclei exhibit "single particle" and others "collective" excitations
    → different models to describe this complementary behavior

This is not an easy question!   The N-N interaction is too complicated to solve in a many-body system: state-of-the-art can go up to A = 3!

First approximation: a square well potential, width approx. equal to nuclear radius R:
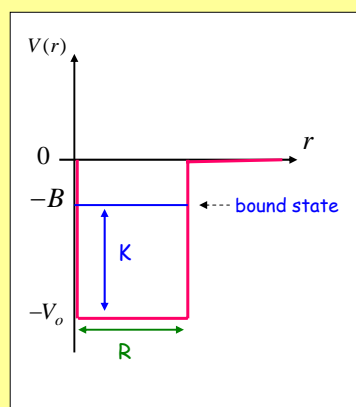
Assume somehow that we can treat the binding of neutrons and protons like electrons in atoms – individual nucleons have wave functions that are eigenstates of some average nuclear potential V(r).
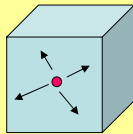
Each nucleon has a **binding energy B** as shown (E = -B)

Kinetic energy K = $(V_0 - B)$

R ≈ 1.2 $A^{1/3}$ fm; most of the wave function is contained inside the well, so this should be approximately the right nuclear size...

Kinetic energy of bound nucleons:

Key point: once we specify the width of the well, the nucleons are confined, and so their kinetic energy is essentially determined by the uncertainty principle:

Simple estimate: Confining box of side 2 fm. $\Delta p_x \, \Delta x \sim \hbar$

$$\bar{p}_x \; = \; 0$$

$$\Delta p_x \; = \; \sqrt{\left\langle (p_x - \bar{p}_x)^2 \right\rangle} \; = \; \sqrt{\left\langle p_x^2 \right\rangle} \; = \; \hbar / \Delta x$$

$$\Rightarrow \; \left\langle p^2 \right\rangle \; = \; 3 \, (\hbar / \Delta x)^2 \; \sim \; 3 \times 10^4 \;\; \mathrm{MeV}^2$$

$$\frac{K}{M} \; = \; \frac{\left\langle p^2 \right\rangle}{2M^2} \; \approx \; 0.015$$

Conclusion: the motion is non relativistic; $K \approx 15$ MeV

Now we can specify the potential parameters:

$V(r)$

0 ———————————————— r

$-B$ ←—— ~ - 3 MeV

$K \sim 15$ MeV

$-V_o$ ←—— ~ - 18 MeV

$R \sim 2$ fm
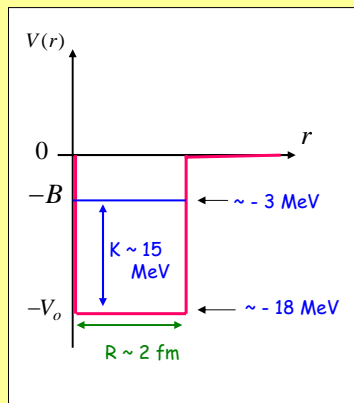
What next?

· We have a complicated system of A nucleons.

· About half of them are protons, so a repulsive (+ve energy) term has to be added to the square well to account for this   (~ few MeV)

How to connect this model to something observable?

Independent particle model:

· Assume independent particle motion in some average nuclear potential V(r) as shown.

· Then we can fill the eigenstates of the potential to maximum occupancy to form a nucleus, as is done with electrons in atoms (to 1st order...)

Connection to average nuclear properties:

- The binding energy of each nucleon, in our model, is a few MeV.

- The potential energy of a bound nucleon is **negative**, by ~ 0.3% of its rest mass energy, which therefore has to show up as a decrease in its mass.

- For A nucleons, the **total binding energy** is:

$$B = \sum_{i=1}^{A} B_i \; = \; \sum_{i=1}^{A} m_i \; - \; M$$

mass of nucleus, M

The average **binding energy per nucleon, B/A**, can be determined from mass data and used to refine a model for V(r); it ranges systematically from about 1 – 9 MeV as a function of mass number for the stable isotopes.

Reference: F&H ch. 16

Atomic Mass Units:

- By convention, we set the mass of the carbon-12 **atom** as a standard.

- Denote atomic masses with a "script" **M,** measured in **atomic mass units,   U**

    M (^{12}C) ≡ 12.0000000000 ..... **U** (exact!)   →   1 **U** = 931.494 MeV (expt.)
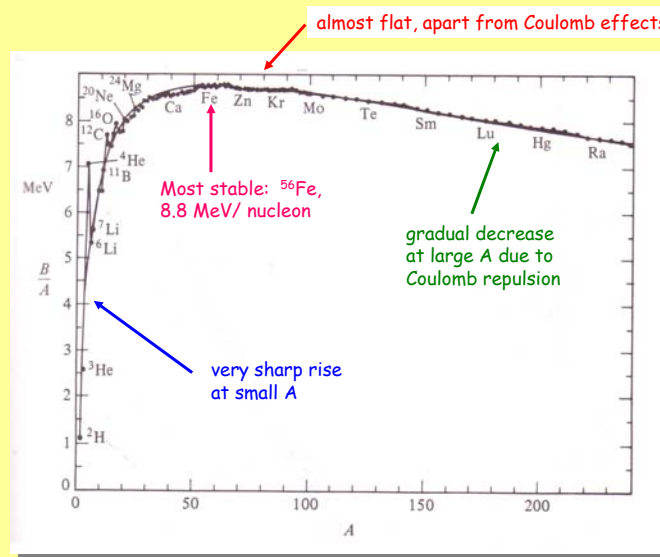
Calculation for carbon-12:

$$m_p = 938.2 \text{ MeV}$$
$$m_n = 939.6 \text{ MeV}$$
$$m_e = 0.511 \text{ MeV}$$

$$6 \times \sum_i m_i = 11{,}269.8 \text{ MeV}$$

$$12\,U \; = \; 11{,}178.0 \text{ MeV}$$

$$B \, (^{12}_{6}C) \; = \; \sum_i m_i - M \; = \; 91.8 \text{ MeV}$$

Binding energy per nucleon in ^{12}C: B/A = 7.8 MeV;

Contrast to the deuteron ^{2}H: B/A = 1.1 MeV

almost flat, apart from Coulomb effects

Most stable: ^{56}Fe,
8.8 MeV/ nucleon
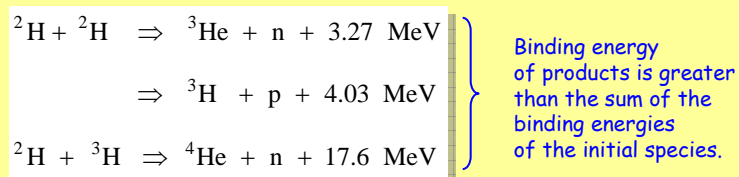
gradual decrease
at large A due to
Coulomb repulsion

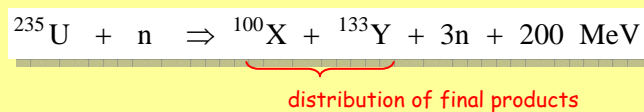very sharp rise
at small A

Greater binding energy implies lower mass, greater stability.

Energy is released when configurations of nucleons change to populate the larger B/A region → nuclear energy generation, e.g.

Fusion reactions at small A release substantial energy **because the B/A curve rises faster than a straight line at small A:**
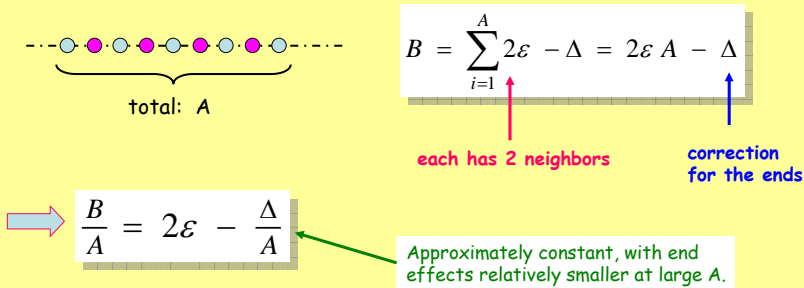
$$^2H + {}^2H \;\Rightarrow\; {}^3He + n + 3.27 \text{ MeV}$$
$$\Rightarrow\; {}^3H + p + 4.03 \text{ MeV}$$
$$^2H + {}^3H \;\Rightarrow\; {}^4He + n + 17.6 \text{ MeV}$$

Binding energy
of products is greater
than the sum of the
binding energies
of the initial species.

Fission reactions at large A release energy because the products have greater binding energy per nucleon than the initial species:

$$^{235}U + n \;\Rightarrow\; {}^{100}X + {}^{133}Y + 3n + 200 \text{ MeV}$$

distribution of final products

A semi-empirical model for nuclear binding energies:

1. Volume and Surface terms:

First consider a 1-dimensional row of nucleons with interaction energy per pair = ε

$$B = \sum_{i=1}^{A} 2\varepsilon - \Delta = 2\varepsilon A - \Delta$$

total: A

each has 2 neighbors

correction for the ends
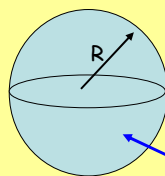
$$\frac{B}{A} = 2\varepsilon - \frac{\Delta}{A}$$

Approximately constant, with end effects relatively smaller at large A.

By analogy, for a 3-d nucleus, **there should be both volume and surface terms with the opposite sign**, the surface nucleons having less binding energy:

$$B = a_V A - a_S A^{2/3} \Rightarrow \frac{B}{A} = a_V - a_S A^{-1/3}$$

---

2. Coulomb term:

for a uniform sphere,

R

total charge + Ze

$$E_{Coul} = \int \frac{q(r)\, dq}{4\pi\varepsilon_o\, r} = \frac{3}{5} \frac{(Ze)^2}{4\pi\varepsilon_o R}$$

This effect increases the total energy and so decreases the binding energy.

Simple model: $\Delta B = - a_C Z^2 A^{-1/3}$

But this is not quite right, because in a sense it includes the Coulomb self energy of a single proton by accounting for the integral from 0 to $r_p \sim 0.8$ fm. The nucleus has fuzzy edges anyway, so we will have to fit the coefficient a_c to mass data.

**Solution: let ΔB scale as the number of proton pairs** and include a term:

$$\Delta B = - a_C Z (Z-1) A^{-1/3} \Rightarrow \frac{\Delta B}{A} = - a_C Z (Z-1) A^{-4/3}$$

3. Symmetry Term:

So far, our formula doesn't account for the tendency for light nuclei to have Z = N. The nuclear binding energy ultimately results from filling allowed energy levels in a potential well V(r).   **The most efficient way to fill these levels is with Z = N:**

<u>Simplest model:</u> identical nucleons as a Fermi gas, i.e. noninteracting spin-$\frac{1}{2}$ particles in a box.   Two can occupy each energy level.   The level spacing ~ 1/A.

A mismatch between Z and N costs an energy price of ΔE at fixed A as shown.
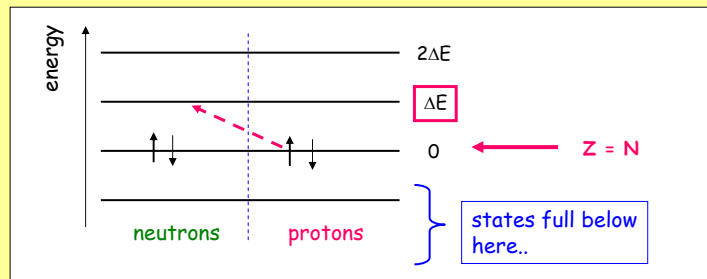
$$\Delta B = -\, a_A \, (Z - N)^2 \, A^{-1} \;=\; -\, a_A \, (A - 2Z)^2 \, A^{-1}$$

---

4. Pairing Term:

Finally, recall from slide 1 that for the case of **even A**, there are 177 stable nuclei with Z and N both even, and only 6 with Z and N both odd.   **Why?**

→ Configurations for which protons and neutrons separately can form pairs must be much more stable.   All the even-even cases have $J^\pi = 0^+$, implying that neutrons and protons have lower energy when paired to total angular momentum zero.

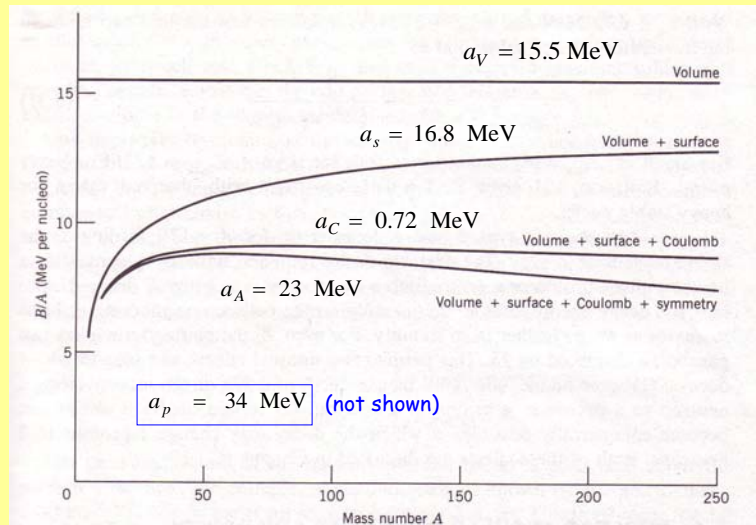Solution: add an empirical **pairing term** to the binding energy formula:

$$\Delta B_{pair} \;\equiv\; \delta \;=\; \begin{bmatrix} +1 \\ 0 \\ -1 \end{bmatrix} a_p \; A^{-3/4}$$

with +1 for even-even, 0 for even-odd, and -1 for odd-odd

Full expression:

$$B(Z, A) \;=\; a_V A - a_S A^{2/3} - a_C \, Z\,(Z-1)\, A^{-1/3} \;-\; a_A\,(A-2Z)^2\,A^{-1} \;+\; \delta$$

13

Fitting of coefficients to data:

$$B(Z,A) = a_V A - a_S A^{2/3} - a_C\, Z\,(Z-1)\,A^{-1/3} - a_A\,(A-2Z)^2\,A^{-1} + \delta$$

$a_V = 15.5$ MeV   Volume

$a_s = 16.8$ MeV   Volume + surface

$a_C = 0.72$ MeV   Volume + surface + Coulomb

$a_A = 23$ MeV   Volume + surface + Coulomb + symmetry

$a_p = 34$ MeV (not shown)

B/A (MeV per nucleon)

Mass number A

One more look at the Binding Energy per Nucleon curve:

Solid line: fit to the semi-empirical formula

almost flat, apart from Coulomb effects

some large oscillations at small mass

Most stable: ^{56}Fe, 8.8 MeV/ nucleon

gradual decrease at large A due to Coulomb repulsion

very sharp rise at small A

$\frac{B}{A}$

A